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Free-electron-laser model without the slowly-varying-envelope approximation

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A mathematical model is presented which describes the evolution of electromagnetic radiation in a free-electron laser (FEL), without any stringent assumptions regarding the envelope of the radiation pulse. The derived set of equations is nearly identical to the traditional set, which needed the assumption of a slowly varying radiation amplitude and phase. Although rapid variations in the radiation envelope do have influence on the dynamics of the electrons, ignoring this fact does not cause excessive errors. Consequently, it is concluded that the region of validity of the traditional FEL equations is much larger than has been realized so far.

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In a free-electron laser (FEL), coherent radiation is produced by a beam of relativistic electrons propagating through a periodic magnetic structure, called the undulator. The first description of the FEL interaction by Madey [1] was based upon quantum mechanics, but it was soon realized that the mechanism is essentially classical. The foundation for classical FEL theory was laid by Colson [2,3], who showed that the exchange of energy between an electron beam and a copropagating radiation field may be modeled in terms of the motion of the individual electrons in a ponderomotive potential, formed by the undulator field and the radiation. This description, similar to the usual description of particle dynamics in linear accelerators, results in a set of first-order differential equations, known as the "Maxwell-pendulum equations." These equations show how individual electrons in the electron beam evolve similarly to pendula in a gravitational field (represented by the radiation amplitude) while the radiation evolution is driven by the longitudinal distribution of the electrons within a radiation wavelength. Colson's approach has since then been followed by many other authors (see, e.g., Refs. [4–7], who contributed to the current status of classical FEL theory.

A characteristic common to all FEL models so far is the assumption that the optical field varies on a scale which is large with respect to a radiation wavelength. This "slowly-varying-envelope approximation" or SVEA [2–8] may be expressed mathematically as

$$\left|\frac{\partial a_r}{\partial z}\right| \ll |k_r a| \ , \tag{1a}$$

$$\left|\frac{\partial a_r}{\partial t}\right| \ll |\omega_r a_r| \ , \tag{1b}$$

where a_r is the (complex) radiation envelope, k_r is the radiation wave number, and $\omega_r = k_r c$, with c the speed of light *in vacuo*. The SVEA was undoubtedly valid for early FEL experiments like that of Elias *et al.* [9], where the amplification rate of the radiation, or the gain per undulator period, was small. More recently, progress in accelerator technology (in particular the invention of the photocathode linac [10]) has made higher beam currents possible, hence increasing the gain per undulator period, which introduces the possibility of higher gradients in the radiation envelope. Moreover, it has been suggested that high beam currents and/or low beam energies may cause phenomena like "strong superradiance" [11] and synchrontron instabilities [12], in which the SVEA may not be valid anymore. Even in the case of low-gain FEL's, however, the SVEA may be violated when the length of the electron pulses approaches the order of the optical wavelength, the way it can happen, e.g., in the Free-Electron Laser for Infrared Experiments (FELIX [13]), producing up to ≈ 100 μm radiation with a pulse length in the order of a picosecond.

In this paper it is shown how the classical set of FEL equations may also be derived *without* the assumption of a slowly varying radiation envelope. Consequently, the Maxwell-pendulum equations have a much larger range of validity than has been realized, and will accurately describe the evolution of radiation envelopes that vary significantly over one radiation wavelength.

The analytical route presented here closely follows previous work by Colson [2,3], except that all derivatives of the fields are explicitly taken into account. All units are in mks. A frame of reference is set up by the orthogonal unit vectors $\mathbf{u}_x$, $\mathbf{u}_y$, and $\mathbf{u}_z$. The electron beam propagates along the positive z axis, in a magnetostatic helical undulator field represented by the vector potential

$$\mathbf{A}_u(z) = \frac{a_u}{\sqrt{2}} \exp(-ik_u z)\mathbf{u} + \text{c.c.} \ , \tag{2}$$

where a_u is real, $\mathbf{u} \equiv (\mathbf{u}_x + i\mathbf{u}_y)/\sqrt{2}$, and k_u is the undulator's wave number. The copropagating radiation field is described by the vector potential

$$\mathbf{A}_r(z,t) = \frac{-i}{\sqrt{2}} \{ \mathbf{u} a_r(z,t) \exp[i(k_r z - \omega_r t)] - \text{c.c.} \} \tag{3}$$

corresponding to the electric field

$$\mathbf{E}_r(z,t) = -\partial \mathbf{A}_r/\partial t = \frac{-i}{\sqrt{2}} \{ \mathbf{u} e_r(z,t) \exp[i(k_r z - \omega_r t)] - \text{c.c.} \} \ . \tag{4a}$$

Substitution of Eq. (4a) into Eq. (3) shows that the complex envelopes a_r and e_r are related by

$$e_r \equiv i\omega_r a_r - \partial a_r/\partial t \ . \tag{4b}$$

Writing the electric field of the radiation and the magnetic field of the undulator in terms of the vector potentials $\mathbf{A}_r$ and $\mathbf{A}_u$ (Lorentz gauge), the evolution of the relativistic momentum $\gamma m \mathbf{v}$ of a single electron in the combined undulator and radiation vector potential $\mathbf{A}_u + \mathbf{A}_r$ may be written as (see, e.g., Ref. [14])

$$D_e(\gamma m \mathbf{v}) = -e\left[-\frac{\partial \mathbf{A}_r}{\partial t} + \mathbf{v}\times[\nabla\times(\mathbf{A}_u+\mathbf{A}_r)]\right] , \tag{5}$$

where the "electron-convective derivative" $D_e \equiv \partial/\partial t + \mathbf{v}\cdot\mathbf{u}_z\partial/\partial z$. In a one-dimensional approach (no x-y dependence) the perpendicular velocity of the electron $\mathbf{v}_\perp = (\mathbf{v}\cdot\mathbf{u}_x, \mathbf{v}\cdot\mathbf{u}_y, 0)$ follows directly by integration of Eq. (5) [2–7]:

$$\mathbf{v}_\perp = \frac{e}{\gamma m}(\mathbf{A}_u + \mathbf{A}_r) \ . \tag{6}$$

This perpendicular velocity and the electron density $n(z,t)$ determine the transverse current density $\mathbf{J} = n(z,t)e\mathbf{v}_\perp$ that drives the optical field according to Maxwell's wave equation (see, e.g., Ref. [14]):

$$\left[\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial z^2}\right]\mathbf{A}_r = \frac{\mathbf{J}}{\epsilon_0} \ . \tag{7}$$

Substitution of Eq. (3) and the transverse current density into Eq. (7), after a slight manipulation of the terms, results in

$$D_r\left[i\omega_r a_r - \left[\frac{\partial a_r}{\partial t}\right]_{\text{i}} + (\tfrac{1}{2}D_r a_r)_{\text{ii}}\right] = \frac{n(z,t)e^2}{2\gamma m \epsilon_0}[(a_r)_{\text{iii}} + i a_u e^{-i\zeta}] \ . \tag{8}$$

(The labels i, ii, and iii will be used to identify the corresponding terms.) In Eq. (8) the "radiation-convective derivative" has been defined as $D_r \equiv \partial/\partial t + c\,\partial/\partial z$, and the electron's "ponderomotive phase" as $\zeta \equiv (k_u + k_r)z - \omega_r t$. The energy change of a single electron interacting with the FEL's electric field $\mathbf{E}_r$ obeys

$$D_e\gamma = \frac{-e}{mc^2}\mathbf{v}_\perp\cdot\mathbf{E}_r = \frac{e}{mc^2}\mathbf{v}_\perp\cdot\frac{\partial \mathbf{A}_r}{\partial t} \ .$$

Substitution of the field (3) and the transverse velocity (6) results in

$$D_e\gamma = \frac{-e^2 a_u}{2i\gamma m^2 c^2}\left\{\left[i\omega_r a_r - \left[\frac{\partial a_r}{\partial t}\right]_{\text{i}}\right]e^{i\zeta} - \text{c.c.}\right\} + \left[\frac{e^2}{2\gamma m^2 c^2}\frac{\partial}{\partial t}|a_r|^2\right]_{\text{iii}} \ . \tag{9}$$

When space-charge interactions in the electron beam are ignored, Eq. (9) may be used to describe the collective behavior of the electrons in the electron beam. Since the electron's longitudinal position may be obtained directly from its energy and the (local) radiation field by $\beta_z^2 = 1 - \gamma^{-2} - \beta_\perp^2$ or, using Eq. (6),

$$\beta_z^2 = 1 - \frac{1}{\gamma^2} - \frac{e^2|\mathbf{A}_u + \mathbf{A}_r|^2}{\gamma^2 m^2 c^2} , \tag{10}$$

the longitudinal electron density $n(z,t)$ and the ponderomotive phase ζ may be obtained from Eq. (10). These two quantities determine the evolution of the radiation by Eq. (8), which closes the system.

At this point we will analyze the set of second-order FEL equations (8)–(10). The terms labeled with iii in (8) and (9) appear from the radiative correction to the perpendicular electron velocity, represented by the $\mathbf{A}_r$ term in Eq. (6). This radiative correction may be neglected under the assumption $|\mathbf{A}_r| \ll |\mathbf{A}_u|$, which is equivalent to the assumption that the magnetic field of the radiation is much weaker than the undulator's magnetic field. The additional neglect of the terms labeled with i and ii, under the slowly-varying-envelope assumptions (1a) and (1b) reduces Eqs. (8) and (9) to the SVEA equations [2–7]:

$$D_r(i\omega_r a_r) = \frac{n(z,t)e^2}{2\gamma m\epsilon_0}(i a_u e^{-i\zeta}) , \tag{11}$$

$$D_e\gamma = \frac{-e^2 a_u}{2i\gamma m^2 c^2}[(i\omega_r a_r)e^{i\zeta} - \text{c.c.}] \ . \tag{12}$$

Using the definition of the complex electric field envelope (4b), the radiation vector potential envelope a_r may be explicitly expressed in terms of e_r:

$$a_r = \sum_{n=0}^{\infty}\frac{1}{(i\omega_r)^{n+1}}\frac{\partial^n e_r}{\partial t^n} \ .$$

Keeping all derivatives of a_r, Eqs. (8) and (9) may therefore also be written as

$$D_r\left[e_r + \tfrac{1}{2}D_r\sum_{n=0}^{\infty}\frac{1}{(i\omega_r)^{n+1}}\frac{\partial^n e_r}{\partial t^n}\right] = \frac{n(z,t)e^2}{2\gamma m\epsilon_0}(i a_u e^{-i\zeta}) , \tag{13}$$

$$D_e\gamma = \frac{-e^2 a_u}{2i\gamma m^2 c^2}(e_r e^{i\zeta} - \text{c.c.}) \ . \tag{14}$$

Except for the term containing the series expansion, these equations have the same form as the SVEA equations (11) and (12). An interesting detail is the electric field envelope, approximated by $e_r \approx i\omega_r a_r$ in the SVEA equations (11) and (12), which now appears in the exact form in Eqs. (13) and (14). The term with the series expansion in Eq. (13) may be neglected when

$$2\omega_r|e_r| \gg |\partial e_r/\partial t + c\,\partial e_r/\partial z| \ .$$

This is an extremely weak condition, which is only violated when the electric field at a fixed point in the radiation pulse exceeds a relative growth of $\exp(2\omega_r t)$, i.e., a power increase of $\exp(2k_r)$ after 1 m of FEL interaction, which corresponds to a factor of more than 10^{100} even in the case of a long FEL wavelength of 10 cm. Hence, neglecting the series expansion in Eq. (13), the full FEL equations (13) and (14) may be written in the same mathematical form as the SVEA equations (11) and (12).

A difference with regard to the SVEA FEL equations is that, although typically $|\mathbf{A}_r| \ll |\mathbf{A}_u|$, the $\mathbf{A}_r$ term in (10) may not be neglected when no SVEA is used. This can be understood by considering the evolution of the electron's longitudinal velocity from Eq. (10). Ignoring the $\mathbf{A}_r$ term in Eq. (10), it is found after differentiation that

$$D_e\beta_z \approx \frac{1-\beta_z^2}{\gamma\beta_z} D_e\gamma \ .$$

Keeping the $\mathbf{A}_r$ term in Eq. (10) brings up an additional term:

$$D_e\beta_z = \frac{1-\beta_z^2}{\gamma\beta_z} D_e\gamma - \frac{e^2}{2\beta_z\gamma^2 m^2 c^2} D_e|\mathbf{A}_r + \mathbf{A}_u|^2 \ . \qquad (15)$$

After substitution of the fields (2) and (3) and assuming "FEL resonance" [2,3], i.e.,

$$\frac{\beta_z}{1-\beta_z} \approx \frac{k_r}{k_u} \ , \qquad (16)$$

Eq. (15) may be written as

$$D_e\beta_z = \frac{-2e^2 a_u k_u}{\gamma^2 m^2 c^2 k_r} \frac{1}{2i} \left\{ \left[i\omega_r a_r - \left[\frac{\partial a_r}{\partial t} \right]_{\mathrm{i}} + \left[\frac{k_r}{2k_u} D_e a_r \right]_{\mathrm{iii}} \right] e^{i\zeta} - \mathrm{c.c.} \right\} \qquad (17)$$

(again, terms proportional to a_r were neglected with respect to terms proportional to a_u). As before, the term labeled with i in Eq. (17) is the non-SVEA term, and the term labeled with iii is the radiative correction resulting from the $\mathbf{A}_r$ term in Eq. (10). Assuming an arbitrary (stationary) radiation envelope $f(z-ct)$, it is readily found that the non-SVEA contribution results in a term

$$cf'(z-ct) \ ,$$

where the prime denotes the operator $\partial/\partial(z-ct)$. Using Eq. (16), it is found that the radiative correction contributes the term (assuming $1-\beta_z \ll 1$)

$$\frac{k_r c}{2k_u} f'(z-ct)(\beta_z - 1) \approx -\frac{c}{2} f'(z-ct) \ ,$$

i.e., the radiative correction amounts to one-half of the term that arises due to the neglect of the SVEA. The physical interpretation of the radiative correction term in (15) is the electron slipping along a non-slowly-varying radiation pulse, which will affect its transverse velocity as seen from Eq. (6). Since the transverse velocity is coupled to the longitudinal velocity as seen from Eq. (10), the longitudinal dynamics will consequently also be affected. Obviously, this effect disappears in the case of a smooth radiation envelope.

In a numerical simulation, inclusion of the effect of the radiation on the transverse motion of the electrons is nontrivial and particularly time-consuming. After discretizing e_r and a_r on a numerical grid, it is possible to obtain a_r from e_r along all the grid points, by solving the set of linear difference equations $e_r^n \equiv i\omega_r a_r^n - (a_r^{n+1} - a_r^{n-1})/2\Delta t$ (where the superscript denotes the ordinal number of the grid points and Δt is the temporal resolution). However, the number of grid points along the pulse, and hence the number of equations to be solved, is usually very large, particularly when several grid points are taken over an interval of one radiation wavelength. From the discussion above, however, it follows that when the term labeled with iii in Eq. (17) is neglected—so that the equation, written in terms of e_r, obtains the same form as the classical SVEA pendulum equation [2,3]—roughly $\frac{2}{3}$ of the radiation envelope derivatives will be taken into account anyway. The induced error is therefore not expected to be excessive.

This statement is confirmed by Fig. 1, in which the evolution of a narrow spike in the radiation pulse is shown for a single-pass, high-gain FEL amplifier. Typical parameters are listed in Table I. The curves represent three different numerical solutions (using a total of 6000 sample electrons): (a) the "classical" SVEA solution, using Eqs. (11) and (12) and taking one radiation grid point per radiation wavelength; (b) the "full" solution, using Eqs. (13) and (14), taking 50 grid points per radiation wavelength and *ignoring* the radiative correction term in (15); and (c) similar to (b), but including the effect of the radiation gradient on the perpendicular electron motion. It is seen how the higher discretization of the radiation field, which is allowed now that the SVEA is dropped, induces a 40% change in the numerical results [curve (a) compared to (b)]. The error due to the neglect of the radiative correction in Eq. (15) is limited to about 20% [curves (b) and (c)].

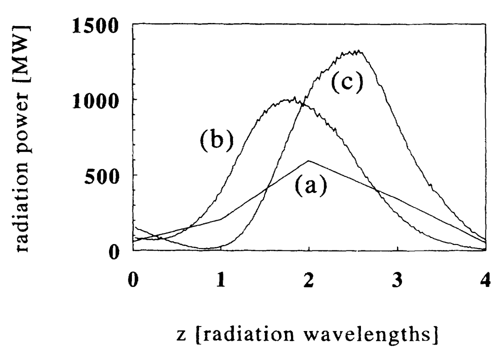

FIG. 1. Spiking envelope of the radiation pulse for a high-gain, single-pass FEL amplifier (violating the assumption of a slowly varying amplitude and phase), obtained from a numerical solution based on Eqs. (11)–(14). The horizontal axis runs along the optical pulse, the vertical axis shows the local radiation power. The curves represent (a) the classical SVEA solution, using Eqs. (11) and (12) and taking 1 radiation sample per radiation wavelength; (b) the full solution, using Eqs. (13) and (14), taking 50 radiation samples per radiation wavelength and *ignoring* the radiative correction in Eq. (17); and (c) identical to (b), but including the radiative correction in Eq. (17). Parameters are listed in Table I.

TABLE I. Physical parameters for the computer simulation shown in Fig. 1.

Parameter	Value
Electron beam	
Energy	5 MeV
Current	100 A
Radius	0.5 mm
Length	16 mm
Undulator	
Peak field	0.2 T
Period	5 cm
No. of periods	40
Radiation	
Wavelength	0.4 mm
Initial intensity	2 kW

It is concluded that the classical SVEA FEL equations may be transformed to a set which is also valid when the radiation envelope changes rapidly on the scale of a radiation wavelength. This transformation simply consists of the substitution of the SVEA approximation for the electric field, in this work represented by $i\omega_r a_r$, by the exact expression $e_r = i\omega_r a_r - \partial a_r/\partial t$. Consequently, both the form and the interpretation of the equations remains identical. This fact implies that the classical SVEA equations have a much larger range of validity than has been realized so far, and may be used to describe all effects that would violate the SVEA, like very short radiation pulses or spiking radiation envelopes. Apparently, in numerical work, it is also allowed to discretize the radiation field on a scale smaller than a radiation wavelength (which is inconsistent with the SVEA), which may have large effects on the obtained results. A modest discrepancy was found concerning the longitudinal motion of the electrons, which will be affected as the electrons slip along a rapidly varying radiation envelope.

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